WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #8 Key

Problem 1. Show that the stationary elastic equation

$$-D(\partial)^T \mathscr{A}(x) D(\partial) u = f \qquad \text{in } \Omega$$

with $a_{11} = a_{22} = a_{33}$, $a_{44} = a_{55} = a_{66} = \mu$, $a_{12} = a_{23} = a_{13} = \lambda = a_{11} - 2a_{44}$, and all other entries of the 6×6 matrix $\mathscr{A}$ being zero, results in the isotropic elastic equation

$$-\nabla \cdot [\mu(\nabla u + \nabla u^T)] - \nabla[\lambda \nabla \cdot u] = f \; .$$

In this context it may be useful to recall the operator

$$D(\partial) = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix} .$$

With the given date the real symmetric 6×6 matrix $\mathscr{A}$ becomes

$$\mathscr{A} = \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

and then

$$\begin{aligned} \mathscr{A} D(\partial) u &= \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \partial_1 u_1 \\ \partial_2 u_2 \\ \partial_3 u_3 \\ \partial_2 u_3 + \partial_3 u_2 \\ \partial_3 u_1 + \partial_1 u_3 \\ \partial_1 u_2 + \partial_2 u_1 \end{bmatrix} \\ &= \begin{bmatrix} (2\mu + \lambda)\partial_1 u_1 + \lambda \partial_2 u_2 + \lambda \partial_3 u_3 \\ \lambda \partial_1 u_1 + (2\mu + \lambda)\partial_2 u_2 + \lambda \partial_3 u_3 \\ \lambda \partial_1 u_1 + \lambda \partial_2 u_2 + (2\mu + \lambda)\partial_3 u_3 \\ \mu(\partial_2 u_3 + \partial_3 u_2) \\ \mu \partial_3 u_1 + \partial_1 u_3) \\ \mu(\partial_1 u_2 + \partial_2 u_1) \end{bmatrix} . \end{aligned}$$

Finally,

$$D(\partial)^T \mathscr{A} D(\partial)u = \begin{bmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} (2\mu+\lambda)\partial_1 u_1 + \lambda\partial_2 u_2 + \lambda\partial_3 u_3 \\ \lambda\partial_1 u_1 + (2\mu+\lambda)\partial_2 u_2 + \lambda\partial_3 u_3 \\ \lambda\partial_1 u_1 + \lambda\partial_2 u_2 + (2\mu+\lambda)\partial_3 u_3 \\ \mu(\partial_2 u_3 + \partial_3 u_2) \\ \mu(\partial_3 u_1 + \partial_1 u_3) \\ \mu(\partial_1 u_2 + \partial_2 u_1) \end{bmatrix}$$

$$= \begin{bmatrix} \partial_1[(2\mu+\lambda)\partial_1 u_1 + \lambda\partial_2 u_2 + \lambda\partial_3 u_3] + \partial_3[\mu(\partial_3 u_1 + \partial_1 u_3)] + \partial_2[\mu(\partial_1 u_2 + \partial_2 u_1)] \\ \partial_2[\lambda\partial_1 u_1 + (2\mu+\lambda)\partial_2 u_2 + \lambda\partial_3 u_3] + \partial_3[\mu(\partial_2 u_3 + \partial_3 u_2)] + \partial_1[\mu(\partial_1 u_2 + \partial_2 u_1)] \\ = \partial_3[\lambda\partial_1 u_1 + \lambda\partial_2 u_2 + (2\mu+\lambda)\partial_3 u_3] + \partial_2[\mu(\partial_2 u_3 + \partial_3 u_2)] + \partial_1[\mu(\partial_3 u_1 + \partial_1 u_3)] \end{bmatrix}$$

$$\nabla[\lambda\nabla\cdot u] + [\partial_1 \quad \partial_2 \quad \partial_3]\mu \begin{bmatrix} 2\partial_1 u_1 & \partial_2 u_1 + \partial_1 u_2 & \partial_1 u_3 + \partial_3 u_1 \\ \partial_2 u_1 + \partial_1 u_2 & 2\partial_2 u_2 & \partial_3 u_2 + \partial_2 u_3 \\ \partial_1 u_3 + \partial_3 u_1 & \partial_3 u_2 + \partial_2 u_3 & 2\partial_3 u_3 \end{bmatrix}$$

$$= \nabla\cdot[\mu(\nabla u + \nabla u^T)] + \nabla[\lambda\nabla\cdot u]$$

Problem 2. a.) Show that the operator

$$P(D) = \frac{1}{4}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)^2$$

considered on the unit disk centered at the origin is not properly elliptic. This is to say that the symbol $p(\xi + in(x)\lambda)$ of the operator on the boundary S^1 with $\xi \perp n(x)$, $\xi \neq 0$ does not decompose into a product of polynomials p_+ and p_- with roots in λ having only positive real part (negative real part) with the degree of p_+ independent of ξ.

Solution. Note that the equation $4P_2(\xi + in(x)\lambda) = -(\xi_1 + in_1\lambda + i\xi_2 - n_2\lambda)^2 = 0$ has the double root

$$\lambda = \frac{\xi_1 + i\xi_2}{n_2 - in_1} = (\xi_1 + i\xi_2)(n_2 + in_1) = \xi_1 n_2 - \xi_2 n_1 \ ,$$

where we used that $\xi_1 n_1 + \xi_2 n_2 = 0$. Here $x \in S^1$. Choosing $x = (1,0)$ implies $n = (1,0)^T$ and hence $\xi = (0,a)$ with $a \in \mathbb{R}\setminus\{0\}$. Note that $\lambda = -a$ which is positive for $a < 0$ and negative for $a > 0$.

b.) Verify that the Dirichlet problem in the unit disk is not regular by constructing an infinite-dimensional space of solutions to the equation $Pu = 0$ with $u|_{S^1} = 0$.

Solution. One verifies that

$$\frac{1}{4}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)^2 = \left(\frac{\partial}{\partial \overline{z}}\right)^2$$

where $z = x + iy$. Furthermore, a holomorphic function f in the unit disk satisfies the equation $\partial f/\partial\overline{z} = 0$ and the space of bounded, holomorphic functions on the the unit disk is infinite-dimensional. Finally, observe that the function

$$u(z,\overline{z}) = (z\overline{z} - 1)f(z) \ ,$$

where f is bounded and holomorphic on the unit disk, satisfies the differential equation $P(D)u = 0$ in the unit disk and homogeneous Dirichlet conditions.

Problem 3. Let $e, h : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be vector-valued functions with three components each and suppose that ε, μ, σ are real symmetric 3×3 matrix functions $\varepsilon, \mu, \sigma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{3\times 3}$ and that ε and μ are uniformly positive definite.

a.) Show that the dynamic Maxwell equations

$$\partial_t(\varepsilon e) - \nabla \times h + \sigma e = f_1 \qquad \partial_t(\mu h) + \nabla \times e = f_2$$

form a symmetric hyperbolic system of order 1 in the sense of Definition 3.1.1.

Solution. Observe that the Maxwell-operator is the 6×6 system

$$P(t,x;D) = \begin{bmatrix} \varepsilon\partial_t & -\nabla\times \\ \nabla\times & \mu\partial_t \end{bmatrix} + \begin{bmatrix} \partial_\varepsilon + \sigma & 0 \\ 0 & 0 \end{bmatrix},$$

where we have used 3×3 blocks. Observe that

$$\operatorname{curl} \mathrm{u} = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix} = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Then, the coefficients are the following 6×6 matrices.

$$A^0(t,x) = \begin{bmatrix} \varepsilon(t,x) & 0 \\ 0 & \mu(t,x) \end{bmatrix}, \qquad A^1(t,x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2(t,x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^3(t,x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These matrices are symmetric.

b.) Can you find conditions on ε and μ which make this system constantly hyperbolic ? Hint: Make use of the solution of the first problem of homework #1.

Solution. Note that

$$[A^0(t,x)]^{-1} \sum_{j=1}^{3} A^j(t,x)\xi_j = \begin{bmatrix} 0 & -\varepsilon^{-1}(t,x) \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \\ \mu^{-1}(t,x) \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} & 0 \end{bmatrix}$$

claim. If ε and μ are scalar, then the matrix above has three double eigenvalues: $0, \sqrt{\varepsilon\mu}|\xi|, -\sqrt{\varepsilon\mu}\xi|$.

Proof. The eigenvalues are the roots in τ of the polynomial

$$\det \begin{bmatrix} \tau\varepsilon(t,x)I_3 & -\begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} & \tau\mu(t,x)I_3 \end{bmatrix}.$$

The determinant can be computed with some effort and it is equal to $\varepsilon\mu\tau^2[\varepsilon\mu\tau^2 - |\xi|^2]^2$. □

In the case of scalar ε and μ we have shown that the eigenvalues are semi-simple. Indeed, since the matrix above is symmetric, the geometric multiplicities need to match the algebraic multiplicities. The Maxwell equations get more complicated if the coefficients are not scalar. With some additional effort one can prove the following statement. The system is constantly hyperbolic if and only if the coefficients μ and ε are linearly dependent.